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SOME OBSERVATIONS ON THE POTENTIAL FUNCTIONS FOR TRANSVERSE ISOTROPY IN THE PRESENCE OF BODY FORCES

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Abstract—In this paper the potential function formulation for a transversely isotropic material is examined in the presence of body force terms. The equations of equilibrium are reduced to Poisson type equations with the body force terms on the right hand side. The potentials are evaluated for a concentrated body force in an infinite material. For a force perpendicular to the isotropic plane, it is shown that some degree of ambiguity exists in the choice of the potential which explains the different forms used by previous researchers. For a point force parallel to the $z = 0$ isotropic plane, the present results for the potentials agree with previous analysis when $z > 0$ but differ for $z < 0$. The ramifications of this on a recent ring loading analysis conducted by the author are discussed and certain confusing features of this recent solution are resolved. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Solutions to fundamental problems in linear elasticity theory (such as a point force) have been obtained by various mathematical methods. The first investigators to derive such solutions (Kelvin, Boussinesq and Cerutti) obtained them by using the theory of the Newtonian Potential and what is called the method of singularities (Love, 1927; Fung, 1965). Other solutions were obtained by what we might refer to today as rather "ad hoc" methods. That is, based on experience and often times clever and ingenious analysis, they were able to come up with a candidate solution (with the correct singular behavior at the point force location and the right behavior at infinity) and verify that the solution was correct. In most instances this verification procedure would be an equilibrium condition which evaluated the stress field and integrated it over a small volume containing the singularity to show that it provided the correct force. A classic example of this technique was the solution presented by Mindlin (1936) who solved the problem of a force below the surface of an isotropic half space. His procedure consisted of taking the Kelvin solution and patching together other nuclei of strain to clear the surface of traction.

Although these different procedures provided us with some of the most important and useful solutions that we have today, they were not within the framework of a robust mathematical method from which solutions to these and to more general loading cases could be easily obtained. It is possible that the lack of general solution techniques for the equations of elasticity is what motivated early researchers (such as Papkovitch, Neuber, Galerkin and Love for example) to develop the potential function methods. A review of some of these formulations can be found in Fung (1965). Here the equations of elasticity are reduced to the equations of potential theory (more specifically Laplace's or Poisson's equation or the bi-harmonic equation). Since solutions and/or some solution methods in potential theory were well known, it was then possible to derive solutions to various problems in a more direct mathematical fashion. This technique was used by Mindlin (1953) to show how potential theory could be used to succinctly obtain his previous solution.

Upon leaving the realm of isotropic materials, the only case in which researchers have successfully applied potential theory is when the material is transversely isotropic. Although it is apparent that investigations into this type of material date to the turn of the century

(Michell, 1900), a potential function formulation for this class of materials was first given by Elliot (1948). Using two potentials ϕ_1 and ϕ_2, Elliot gave the solution for a point force (perpendicular to the isotropic $z = 0$ plane) in an infinite body in terms of the functions $\phi_i = \ln\{(R_i+z_i)/(R_i-z_i)\}$, $R_i = \sqrt{r^2+z_i^2}$ and $z_i = z/\sqrt{v_i}$ with v_i being a complex constant. In his initial formulation he did not include body forces and deduces that the potential functions ϕ_i should be harmonic in the three variables x, y and z_i. However it is apparent that his choice for the potential in the point force solution is not harmonic in either the region $z > 0$ or $z < 0$, both regions containing no body force. In particular the above function formally satisfies Laplace's equation but it is not harmonic along the entire z axis. The behavior of these functions has been known for a long time. For example Love (1927) shows that the displacement vector in an isotropic body corresponding to a line of centers of dilatation uniformly distributed along the negative z axis is proportional to the gradient of the function $\ln(R+z)$.

Pan and Chou (1976) note other previous solutions to point force loading in a transversely isotropic full space and themselves present a general solution (as they claim valid for any "stable" transversely isotropic material) when a force perpendicular or parallel to the isotropic planes is considered. Their potential formulation is slightly different from Elliot's analysis and it seems to be more complicated. Their "ad hoc" method considers the potentials to contain terms like Elliot's as well as additional terms such as R_i for a perpendicular force and xz_iR_i/r^2 for a tangential force. Again the introductory analysis indicates that the potentials should be harmonic functions since no body forces were included, whereas their potentials are also not harmonic in regions where the body force is zero.

Similar to Elliot's formulation, Fabrikant (1989) provided a quite concise potential function solution for transversely isotropic materials. It included three potentials as Elliot's formulation was incomplete and not capable of handling general loading parallel to the isotropic planes (Green and Zerna, 1968). Fabrikant's formulation provides many simplifications to the cumbersome combinations of elastic constants that appear in these equations which escaped previous researchers. Furthermore he gave the simplest forms to the point force potentials for a force in an infinite body. For a perpendicular force the potentials were in terms of $\ln(R_i+z_i)$ only while for a tangential force the function $z_i\ln(R_i+z_i)-R_i$ was used. Again his introductory formulation did not include body forces and thus, indicated that the potentials were harmonic but the above functions are not harmonic along the negative z axis.

These previous analysis lead to some confusing aspects of the point force solutions for transverse isotropy. First, why do similar formulations (such as that by Elliot and Fabrikant) use different potential functions? Secondly, why do the formulations in the absence of body forces lead these investigators to indicate that the potentials are harmonic in a region when the point force potentials themselves are not harmonic in a certain part of these regions. These anomalies can lead to quite interesting solution behaviors when using these point force Green's functions to solve distributed loading problems. For example, in a recent paper Hanson and Wang (1997) derived the solutions for concentrated ring loading in a transversely isotropic material. The load was distributed along the circumference of a circle with radius $\rho = a$ in a cylindrical coordinate system (ρ, ϕ, z). Fabrikant's potentials were used to perform the integrals. Integrating the ρ derivatives of the potentials around the circle led to functions which had a discontinuity along the semi-infinite cylinder $z < 0$, $\rho = a$. Second and higher ρ derivatives of the potentials also produced additional Dirac delta functions of the form $[1-H(z)]\delta(\rho-a)/\rho$ which the authors did not include since they did not directly contribute to the elastic field (they are essentially zero everywhere except along the semi-infinite cylinder $z < 0$, $\rho = a$). Although they showed that the discontinuous terms cancelled out when the elastic field was evaluated, their presence at all raised questions in the mind of the author. To further confuse the issue, when the Dirac delta functions were included as they should have been, they canceled out of the elastic field for a perpendicular ring load but not for a ring load in the plane of isotropy. It seemed odd that an elastic field for a ring load would contain explicit Dirac delta functions.

Upon further study, it became apparent that the only way to clarify these confusing aspects was a fundamental investigation of the basic equations while retaining the body force terms. Here the intention is to provide the basic equations in the spirit of a Papkovitch–Neuber formulation for a transversely isotropic material where the potentials can be found from a Poisson type equation with the body force terms on the right hand side. Such an analysis is presented below and provides all of the information necessary to clarify the questions and confusion noted above. As a final comment it is mentioned that the point force Green's functions for transverse isotropy are included as a special case in a new analysis by Ting and Lee (1997) for general anisotropic materials. However no consideration was given to the determination of the potential functions which is the main interest presently.

2. FORMULATION FOR TRANSVERSE ISOTROPY

In this section the potential function formulation developed by Elliot (1948) for transverse isotropy is given. Although it is well established and has been reported many times elsewhere, some pertinent equations are needed for the present discussion. The notation used by Fabrikant (1989) is adopted since, in the present author's opinion, it is the most convenient in terms of simplifying the parameters that are given as combinations of the elastic constants and in simplifying the equations by using a complex notation. A rectangular coordinate system will be used in which z is taken perpendicular to the horizontal isotropic plane (positive z is directed down). The displacement components are introduced as u, v and w in the x, y and z directions, respectively. The stresses are given in terms of the strains as

$$\sigma_{xx} = A_{11}\frac{\partial u}{\partial x}+(A_{11}-2A_{66})\frac{\partial v}{\partial y}+A_{13}\frac{\partial w}{\partial z}$$

$$\sigma_{yy} = (A_{11}-2A_{66})\frac{\partial u}{\partial x}+A_{11}\frac{\partial v}{\partial y}+A_{13}\frac{\partial w}{\partial z}$$

$$\sigma_{zz} = A_{13}\frac{\partial u}{\partial x}+A_{13}\frac{\partial v}{\partial y}+A_{33}\frac{\partial w}{\partial z}$$

$$\tau_{xz} = A_{44}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right),\quad \tau_{yz} = A_{44}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)$$

$$\tau_{xy} = A_{66}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \qquad (1)$$

The standard procedure is now to substitute the above relations into the equations of equilibrium. This has been done in the two references noted above in the absence of body forces. For present purposes we must include the body force terms and the three displacement equations of equilibrium become

$$A_{44}\Delta w+A_{33}\frac{\partial^2 w}{\partial z^2}+\frac{1}{2}(A_{13}+A_{44})\frac{\partial}{\partial z}(\bar{\Lambda}u^c+\Lambda\bar{u}^c)+B_z = 0$$

$$(A_{13}+A_{44})\Lambda\frac{\partial w}{\partial z}+A_{44}\frac{\partial^2 u^c}{\partial z^2}+\frac{1}{2}(A_{11}+A_{66})\Delta u^c+\frac{1}{2}(A_{11}-A_{66})\Lambda^2\bar{u}^c+B^c = 0 \qquad (2)$$

The complex in-plane displacement is defined as $u^c = u+iv$, B_x, B_y and B_z are the body forces in the coordinate directions and $B^c = B_x+iB_y$ is the complex body force. An overbar indicates complex conjugation and the differential operators are defined as

$$\Lambda = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}, \quad \Lambda\bar{\Lambda} = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{3}$$

The second equilibrium equation above is actually two equations if one separates real and imaginary parts.

A potential function solution to the above equations in the absence of body forces was first given by Elliot (1948). His solution contained two potential functions and he used them to solve for a point force in an infinite space when the force was in the z direction. Here we refer to this as normal loading since the force is perpendicular (normal) to the isotropic planes. Elliot's potential function solution is also reported in Green and Zerna (1968) where a third potential function was added. This third function corresponds to axisymmetric torsional loading in the x, y plane and is needed to solve problems of tangential loading (when the loading is directed parallel to the isotropic planes).

In Fabrikant's notation, the displacements are written in terms of the three potentials as

$$u^c = \Lambda(F_1 + F_2 + iF_3), \quad w = m_1\frac{\partial F_1}{\partial z} + m_2\frac{\partial F_2}{\partial z} \tag{4}$$

When these are substituted into the displacement equations of equilibrium, three partial differential equations are obtained for the potentials. These are

$$\frac{\partial}{\partial z}\left[A_{33}\sum_{j=1}^{2}\frac{m_j}{\gamma_j^2}L_jF_j\right] + B_z = 0 \tag{5}$$

$$\Lambda\left[A_{11}\sum_{j=1}^{2}L_jF_j + iA_{66}L_3F_3\right] + B^c = 0 \tag{6}$$

The differential operators L_j are defined as

$$L_j = \Delta + \gamma_j^2\frac{\partial^2}{\partial z^2}, \quad j = 1, 2, 3 \tag{7}$$

The constant γ_3 is a positive real number given as $\gamma_3^2 = A_{44}/A_{66}$ while $\gamma_j = \sqrt{n_j}, j = 1, 2$ and n_j are the two (real or complex conjugate) roots of

$$A_{11}A_{44}n_j^2 + [A_{13}(A_{13} + 2A_{44}) - A_{11}A_{33}]n_j + A_{33}A_{44} = 0 \tag{8}$$

The real or complex conjugate constants m_j are related to $\gamma_j^2 = n_j$ as

$$m_j = \frac{A_{11}\gamma_j^2 - A_{44}}{A_{13} + A_{44}} = \frac{(A_{13} + A_{44})\gamma_j^2}{A_{33} - \gamma_j^2 A_{44}}, \quad j = 1, 2 \tag{9}$$

This last equation can be rearranged as

$$\gamma_j^2 = \frac{m_j A_{33}}{A_{13} + (1 + m_j)A_{44}} = \frac{m_j A_{13} + (1 + m_j)A_{44}}{A_{11}} \tag{10}$$

As discussed by Hanson and Wang (1997), either γ_1, γ_2 are both positive real or they are complex conjugate with positive real parts. Also since $m_1 m_2 = 1$ (Fabrikant, 1989), complex conjugate m_j imply they each have a unit modulus.

As this point it is noticed that in a region absent of body forces, a solution to the equilibrium equations can be simply given as

$$L_jF_j = 0, \quad j = 1,2,3 \tag{11}$$

Functions which satisfy the above equation will be termed harmonic. By introducing the new variable $z_j = z/\gamma_j$ the operator L_j becomes the Laplacian in the variables x, y and z_j. It is natural to assume this equation should be true and it has been reported in Elliot (1948) and Fabrikant (1989), for example. This type of assumption would be consistent with isotropic potential function formulations which have been long known. However it will presently be shown that this equation need not be true in a region devoid of body force. It might be true that if the body force vanishes everywhere in the body, then the potentials are harmonic. If the body force does not vanish at some point, it will be shown that the potentials are not necessarily harmonic at other points, where the body force is zero.

3. NORMAL LOADING

As a starting point normal loading is considered and the only non-zero body force is $B_z(B^c = 0)$. First consider eqn (6) where the term in brackets must be a function of z only. A valid solution (not necessarily the most general) to these equations is to take $F_3 = 0$ (based on known solutions) and to take the function of z to be zero. The equations become

$$\frac{\partial}{\partial z}\left[A_{33}\sum_{j=1}^{2}\frac{m_j}{\gamma_j^2}L_jF_j\right] = -B_z \tag{12}$$

$$L_1F_1 + L_2F_2 = 0 \tag{13}$$

Two choices are now apparent. The first equation could be integrated to eliminate the z derivative and arrive at two equations for L_1F_1 and L_2F_2. This clearly illustrates that these quantities depend on the integral of the body force rather than the body force itself. The integral of the body force may be non-zero in a region where the body force itself vanishes. As a simple example of this consider the arbitrary body force distribution concentrated in the $z = 0$ plane as $B_z = f(x,y)\delta(z)$ where $\delta(z)$ is the Dirac delta function. Obviously the integral of this would be $f(x,y)H(z)$ plus a function independent of z, where $H(z)$ is the Heaviside function. Thus, the integral of the body force could be non-zero in either $z > 0$ or $z < 0$ (or both) depending on how the arbitrary function was chosen. Thus the potential functions are not necessarily harmonic in regions of zero body force. In fact it is easy to see from eqn (13) that the functions F_j can be non-harmonic, it is only required that their non-harmonicities cancel when added together.

Following the alternative procedure of applying a z derivative to the second equation leads to

$$\frac{m_1}{\gamma_1^2}L_1F_1' + \frac{m_2}{\gamma_2^2}L_2F_2' = -\frac{B_z}{A_{33}} \tag{14}$$

$$L_1F_1' + L_2F_2' = 0 \tag{15}$$

where F_j' denotes $\partial F_j/\partial z$. These equations are easily solved to give

$$L_1F_1' = -\frac{B_z}{A_{44}(m_1-m_2)}, \quad L_2F_2' = \frac{B_z}{A_{44}(m_1-m_2)} \tag{16}$$

where the following identity has been used

$$\frac{\gamma_1^2\gamma_2^2}{(m_2\gamma_1^2-m_1\gamma_2^2)A_{33}} = -\frac{1}{A_{44}(m_1-m_2)} \tag{17}$$

It is now clear that the functions F_j' are harmonic in regions of zero body force.

As an example consider a point force of magnitude P applied in the z direction at the origin of a cylindrical coordinate system (ρ, ϕ, z). The equations for F_j' become

$$L_jF_j' = \frac{(-1)^j}{A_{44}(m_1-m_2)}\frac{P}{2\pi\rho}\delta(\rho)\delta(z), \quad j = 1,2 \tag{18}$$

Introducing the variable z_j and using the full space Green's function, it is well known that

$$\left(\Delta+\frac{\partial^2}{\partial z_j^2}\right)\left(-\frac{1}{4\pi R_j}\right) = \frac{1}{2\pi\rho}\delta(\rho)\delta(z_j) \tag{19}$$

where $R_j = \sqrt{\rho^2+z_j^2}$. Now using the result $\delta(z_j) = \gamma_j\delta(z)$ if γ_j is a real positive number, it is possible to write down the solution for F_j' as

$$F_j' = \frac{(-1)^{j+1}P}{4\pi A_{44}(m_1-m_2)\gamma_j}\frac{1}{R_j} \tag{20}$$

This equation can be integrated with respect to z to give

$$F_j = \frac{(-1)^{j-1}P}{4\pi A_{44}(m_1-m_2)}[\ln(R_j+z_j)+h(\rho)] \tag{21}$$

where $h(\rho)$ is an arbitrary function. This is the form reported by Fabrikant (1989) with $h(\rho) = 0$. Observe that the function $\ln(R_j+z_j)$ is harmonic for all $z > 0$ but not harmonic for all $z < 0$. Actually it will be shown below that $L_j\ln(R_j+z_j) = -(2/\rho)\delta(\rho)[-1+H(z)]$ and is not harmonic along the negative z axis. Choosing $h(\rho) = -\ln(\rho^2)$ gives the form

$$F_j = -\frac{(-1)^{j+1}P}{4\pi A_{44}(m_1-m_2)}\ln(R_j-z_j) \tag{22}$$

which is harmonic for all $z < 0$ but not harmonic along the positive z axis. Here the relation

$$\ln(R_j+z_j)+\ln(R_j-z_j) = \ln(R_j^2-z_j^2) = \ln(\rho^2) \tag{23}$$

was used. Considering only the $h(\rho)$ term, it is easily shown from eqn (4) that the displacement field resulting from any choice of $h(\rho)$ will be identically zero everywhere (at every point). Also the $h(\rho)$ term leads to a zero body force everywhere which can be shown from eqns (5) and (6). Hence there is some degree of non-uniqueness in the potential functions. This explains why different forms to the potentials for normal loading produce identical elastic fields.

The above derivation of the potential functions is correct for material combinations which give real (and hence positive) γ_j. However when γ_j are complex conjugate it becomes more difficult to apply generalized function theory since it is not clear to the author how to interpret $\delta(z_j)$ for complex z_j. Of course one could just assume that the results hold for complex γ_j, or now having the potentials one can verify that it represents a point force for any γ_j by finding the stress components and integrating. A different approach is presently adopted in which equations (12) and (13) are solved using the Fourier and Hankel transforms. First this will allow all derivations to be mathematically correct for any γ_j, real or complex. Secondly it will provide a simple way to determine L_jF_j which would be more difficult to otherwise evaluate. Finally it will provide an independent confirmation to the results recently derived by Hanson and Wang (1997) for ring loading.

4. TRANSFORM ANALYSIS

Now the above result for complex γ_j will be established. Additionally it will allow the easy evaluation of L_jF_j for the point force potentials. An axisymmetric distribution of body forces will be assumed. The non-axisymmetric case can be handled with the addition of a Fourier series. A Fourier transform in the z direction will be used coupled with a Hankel transform of order 0 in the radial coordinate. For the body force B_z the combined Fourier and Hankel transform pair is taken as

$$B_z(\rho, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^{\infty} \eta \tilde{B}_z(\eta, \xi) J_0(\eta\rho)\,d\eta\, e^{-i\xi z}\,d\xi$$

$$\tilde{B}_z(\eta, \xi) = \int_0^{\infty} \rho J_0(\eta\rho)\int_{-\infty}^{\infty} B_z(\rho, z)\, e^{i\xi z}\,dz\,d\rho \tag{24}$$

where $\tilde{F}_j(\eta, \xi)$ denotes the double transform of the potentials.

First the Fourier transform is taken of equations (12) and (13) and then the Hankel transform in applied. This leads to the general result

$$\tilde{F}_j(\eta, \xi) = \frac{(-1)^{j+1} i\tilde{B}_z(\eta, \xi)}{\xi(\eta^2+\gamma_j^2\xi^2)A_{44}(m_1-m_2)} \tag{25}$$

where equation (17) was used.

4.1. *Point body force*

For a point force of magnitude P at the origin $\tilde{B}_z(\eta, \xi) = P/(2\pi)$. Substituting this into the above equation and taking the inverse double transform, the potentials become

$$F_j(\rho, z) = \frac{(-1)^{j+1}P}{2\pi^2 A_{44}(m_1-m_2)\gamma_j^2}\int_0^{\infty}\frac{\sin(\xi z)}{\xi}\int_0^{\infty}\frac{\eta J_0(\eta\rho)\,d\eta}{\left(\dfrac{\eta^2}{\gamma_j^2}+\xi^2\right)}d\xi \tag{26}$$

The Fourier integral can be evaluated first as (Erdélyi *et al.*, 1954a)

$$\int_0^{\infty}\frac{\sin(\xi z)}{\xi\left(\dfrac{\eta^2}{\gamma_j^2}+\xi^2\right)}d\xi = \frac{\pi\gamma_j^2\,\mathrm{sgn}(z)}{2\eta^2}\left[1-e^{-\eta\frac{|z|}{\gamma_j}}\right] \tag{27}$$

where sgn(z) is the sign function. The above result is valid for $Re(\eta/\gamma_j) > 0$ which is valid since $Re(\gamma_j) > 0$ for any transversely isotropic material. The Hankel integral is evaluated as (Erdélyi *et al.*, 1954b)

$$\int_0^{\infty}\frac{1}{\eta}J_0(\eta\rho)\left[1-e^{-\eta\frac{|z|}{\gamma_j}}\right]d\eta = \sinh^{-1}\left(\frac{|z|}{\rho\gamma_j}\right) \tag{28}$$

which is valid for $Re(|z|/\gamma_j) > 0$ and thus, also always true. The potentials then become

$$F_j(\rho, z) = \frac{(-1)^{j-1} P \operatorname{sgn}(z)}{4\pi A_{44}(m_1 - m_2)} \ln\left[\frac{\frac{|z|}{\gamma_j} + R_j}{\rho}\right] \tag{29}$$

This can be rewritten as

$$F_j(\rho, z) = \frac{(-1)^{j+1} P}{4\pi A_{44}(m_1 - m_2)} \ln\left[\frac{R_j + z_j}{\rho}\right] \tag{30}$$

This is identical to eqn (21) with $h(\rho) = -\ln(\rho)$. Of course the $\ln(\rho)$ term gives no displacement and could be discarded. However since the transform results above will be used below, this term will be retained automatically.

Now consider the harmonic behavior of the above potential. Applying the differential operator L_j to $F_j(\rho, z)$ in the form of the double integral gives

$$\begin{aligned} L_j F_j(\rho, z) &= L_j \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \eta \tilde{F}_j(\eta, \xi) J_0(\eta\rho)\, \mathrm{d}\eta\, e^{-i\xi z}\, \mathrm{d}\xi \\ &= \frac{-1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \eta \tilde{F}_j(\eta, \xi)(\eta^2 + \gamma_j^2 \xi^2) J_0(\eta\rho)\, \mathrm{d}\eta\, e^{-i\xi z}\, \mathrm{d}\xi \end{aligned} \tag{31}$$

Substituting for $\tilde{F}_j(\eta, \xi)$ from eqn (25) and remembering $\tilde{B}_z(\eta, \xi) = P/(2\pi)$ provides

$$L_j F_j(\rho, z) = \frac{(-1)^j P i}{4\pi^2 A_{44}(m_1 - m_2)} \int_{-\infty}^{\infty} \frac{e^{-i\xi z}}{\xi}\, \mathrm{d}\xi \int_0^{\infty} \eta J_0(\eta\rho)\, \mathrm{d}\eta \tag{32}$$

Although neither integral exists in the ordinary sense, they are both well defined in terms of generalized function theory giving

$$L_j F_j(\rho, z) = \frac{(-1)^{j+1} P}{4\pi A_{44}(m_1 - m_2)} \frac{\delta(\rho)}{\rho} [1 - 2H(z)] \tag{33}$$

Obviously these potentials are not harmonic in either $z < 0$ or $z > 0$ because of the source distribution along the z axis. Again the sources are not body forces but rather result from an integration of the body forces. If one discards the $\ln(\rho)$ term in the potentials $F_j(\rho, z)$ given in eqn (30) (making it the potential given by Fabrikant, 1989 and denoted by a superscript F), the result is

$$L_j F_j^F(\rho, z) = \frac{(-1)^{j+1} P}{2\pi A_{44}(m_1 - m_2)} \frac{\delta(\rho)}{\rho} [1 - H(z)] \tag{34}$$

which is harmonic for $z > 0$ but not along the negative z axis. Note that the above two equations reveal either form to the potential satisfies eqns (12) and (13).

4.2. *Ring normal loading*

As a second example consider a body force uniformly distributed along the circle $\rho = a$ in the $z = 0$ plane. This will be termed ring normal loading. The present transform analysis implies the potential in eqn (30) will be used. If Q denotes the body force intensity per unit circumferential length, the body force and its double transform are

$$B_z(\rho, z) = \frac{Qa}{\rho}\delta(\rho - a)\delta(z), \quad \tilde{\tilde{B}}_z(\eta, \xi) = QaJ_0(\eta a) \tag{35}$$

Substituting this result into eqn (25) and then taking the inverse double transform, the potentials become

$$F_j(\rho, z) = \frac{(-1)^{j+1}Qa}{\gamma_j^2 \pi A_{44}(m_1 - m_2)} \int_0^\infty \eta J_0(\eta a) J_0(\eta\rho) \int_0^\infty \frac{\sin(\xi z)\,d\xi}{\xi\left(\frac{\eta^2}{\gamma_j^2} + \xi^2\right)} d\eta \tag{36}$$

The Fourier integral is the same as before leading to

$$F_j(\rho, z) = \frac{(-1)^{j+1}Qa\,\mathrm{sgn}(z)}{2A_{44}(m_1 - m_2)} \int_0^\infty \frac{1}{\eta} J_0(\eta a) J_0(\eta\rho) \left[1 - e^{-\eta\frac{|z|}{\gamma_j}}\right] d\eta \tag{37}$$

Notice that the term corresponding to the 1 in the brackets is independent of z and hence will give no displacement but is needed for convergence of the integral at $\eta = 0$ and will be retained.

Integrals of the above type were first systematically evaluated by Eason *et al.* (1955) in terms of complete elliptic integrals of the first and second kinds and Heuman's Lambda Function. They were recently put in a more convenient form by Hanson and Puja (1997) through using different parameters and the complete elliptic integral of the third kind. Although the above integral has yet to be evaluated, all of its derivatives are known. Here the notation of Eason *et al.* (1955) is used to denote the integral as

$$I(\mu, v; \lambda) = \int_0^\infty \xi^\lambda J_\mu(a\xi) J_v(\rho\xi)\, e^{-\xi z}\, d\xi \tag{38}$$

The results in Hanson and Puja (1997) re-evaluated the above integral for z real and hence positive. These evaluations will also be valid for z replaced by $|z|/\gamma_j$ which always has a positive real part. The notation $I_{|j|}(\mu, v; \lambda)$ will be used for z replaced by $|z|/\gamma_j$ in the above integral and also in its evaluation in terms of combinations of complete elliptic integrals. The notation $I_j(\mu, v; \lambda)$ will only represent the combination of complete elliptic integrals that $I_j(\mu, v; \lambda)$ is evaluated into when z is replaced by z_j in these formulas. This is an important distinction since the integral above is not defined when z is replaced by $z_j = z/\gamma_j$ and z is in the region $z < 0$ since in that region $Re(z_j) < 0$. However these combinations of elliptic integrals are still valid functions for z replaced by z_j when $z < 0$.

The ρ derivative of the potential from eqn (37) is given as

$$\frac{\partial F_j(\rho, z)}{\partial \rho} = \frac{(-1)^{j+1}Qa\,\mathrm{sgn}(z)}{2A_{44}(m_1 - m_2)} \left\{ I_{|j|}(0, 1; 0) - \int_0^\infty J_0(\eta a) J_1(\eta\rho)\, d\eta \right\} \tag{39}$$

The function $I(0, 1; 0)$ is given by Hanson and Puja (1997) in the form

$$I(0, 1; 0) = \frac{1}{\rho}\left\{1 - \frac{2z}{\pi l_2}\Pi(n, k)\right\} \tag{40}$$

where $\Pi(n, k)$ is the complete elliptic integral of the third kind and the parameters are defined as

$$k = \frac{l_1}{l_2}, \quad n = \frac{l_1^2}{\rho^2} \tag{41}$$

with l_1, l_2 defined by Fabrikant (1989) as

$$l_1 = \tfrac{1}{2}[\sqrt{(\rho+a)^2+z^2} - \sqrt{(\rho-a)^2+z^2}] \quad l_2 = \tfrac{1}{2}[\sqrt{(\rho+a)^2+z^2} + \sqrt{(\rho-a)^2+z^2}] \tag{42}$$

It is easy to see that $l_1 < l_2$ while it can also be shown that $l_1 < \rho$. The parameters l_1, l_2, k and n are all functions of z^2 and hence the absolute value is of no consequence in these parameters. They are denoted as l_{1j}, l_{2j}, k_j and n_j with the substitution $z \to z_j$.

It is easy to show that

$$\begin{aligned} \operatorname{sgn}(z) I_{|j|}(0,1;0) &= \frac{\operatorname{sgn}(z)}{\rho} - \frac{2\operatorname{sgn}(z)|z|}{\pi \rho l_{2j} \gamma_j} \Pi(n_j, k_j) \\ &= \left[\frac{1}{\rho} - \frac{2z_j}{\pi \rho l_{2j}} \Pi(n_j, k_j)\right] - \frac{1}{\rho} + \frac{\operatorname{sgn}(z)}{\rho} \\ &= I_j(0,1:0) + \left\{0, z > 0; -\frac{2}{\rho}, z < 0\right\} \end{aligned} \tag{43}$$

Furthermore it is well known that

$$\int_0^\infty J_0(\eta a) J_1(\eta \rho)\, \mathrm{d}\eta = \frac{1}{\rho} H(\rho - a) \tag{44}$$

Thus, the final form for the ρ derivative can be written as

$$\frac{\partial F_j(\rho, z)}{\partial \rho} = \frac{(-1)^{j+1} Qa}{2A_{44}(m_1 - m_2)} \left[I_j(0,1;0) + \left\{ -\frac{H(\rho-a)}{\rho}, z > 0; \frac{H(\rho-a)-2}{\rho}, z < 0 \right\} \right] \tag{45}$$

Further derivatives are easily found. For example applying two ρ derivatives to eqn (37) provides

$$\frac{\partial^2 F_j(\rho, z)}{\partial \rho^2} = \frac{(-1)^j Qa \operatorname{sgn}(z)}{2A_{44}(m_1 - m_2)} \left\{ \int_0^\infty \eta J_0(\eta a) J_0(\eta \rho) \left[1 - e^{-\eta \frac{|z|}{\gamma_j}}\right] \mathrm{d}\eta - \frac{1}{\rho} \int_0^\infty J_0(\eta a) J_1(\eta \rho) \left[1 - e^{-\eta \frac{|z|}{\gamma_j}}\right] \mathrm{d}\eta \right\} \tag{46}$$

This can be written as

$$\begin{aligned} \frac{\partial^2 F_j(\rho, z)}{\partial \rho^2} &= \frac{(-1)^j Qa \operatorname{sgn}(z)}{2A_{44}(m_1 - m_2)} \left\{ \frac{1}{\rho} \delta(\rho - a) - I_{|j|}(0,0;1) \right\} - \frac{1}{\rho} \frac{\partial F_j(\rho, z)}{\partial \rho} \\ &= \frac{(-1)^{j+1} Qa}{2A_{44}(m_1 - m_2)} \left[\begin{array}{l} -\frac{\operatorname{sgn}(z)}{\rho} \delta(\rho - a) + I_j(0,0;1) - \frac{1}{\rho} I_j(0,1;0) \\ + \left\{ \frac{H(\rho-a)}{\rho^2}, z > 0; \frac{2 - H(\rho-a)}{\rho^2}, z < 0 \right\} \end{array} \right] \end{aligned} \tag{47}$$

which reveals that the second ρ derivative produces a Dirac delta function. Applying two z derivatives to eqn (36) and evaluating the resulting integrals also yields

$$\frac{\partial^2 F_j(\rho,z)}{\partial z^2} = \frac{(-1)^j Qa\,\mathrm{sgn}(z)}{2A_{44}(m_1-m_2)\gamma_j^2} I_{|j|}(0,0;1) = \frac{(-1)^j Qa}{2A_{44}(m_1-m_2)\gamma_j^2} I_j(0,0;1) \tag{48}$$

Using eqns (45), (47) and (48) it is easy to establish

$$L_j F_j(\rho,z) = \frac{(-1)^j Qa\,\mathrm{sgn}(z)}{2A_{44}(m_1-m_2)} \frac{1}{\rho}\delta(\rho-a), \quad L_j F_j'(\rho,z) = \frac{(-1)^j Qa}{A_{44}(m_1-m_2)} \frac{1}{\rho}\delta(\rho-a)\delta(z) \tag{49}$$

where the second equation is obtained by differentiating the first with respect to z and also follows directly from eqns (16) and (35). The first equation above could likewise have been obtained from eqns (25), (31) and (35). The above equation reveals that the individual F_j are not harmonic but differ only by a negative sign so as to satisfy eqn (13). The z derivatives of these potentials obviously also satisfy eqn (12) for the body force in eqn (35).

4.3. *Comparison with Hanson and Wang*

Ring normal loading in a full space was one of the cases evaluated by Hanson and Wang (1997). They used the point force potential given by Fabrikant (1989) (eqn (21) above with $h(\rho) = 0$). Their ring loading potential was

$$F_j^H(\rho,z) = \frac{(-1)^{j+1} Qa}{4\pi A_{44}(m_1-m_2)} \psi(\rho,z_j), \quad \psi(\rho,z_j) = \int_0^{2\pi} \ln\,[R_j+z_j]\,\mathrm{d}\phi_0 \tag{50}$$

where the superscript H is used to differentiate it from $F_j(\rho,z)$ in the previous section and $R_j^2 = \rho^2+a^2-2a\rho\cos(\phi_0)+z_j^2$. From eqn (30) the potential used in the transform analysis of the previous section corresponds to

$$F_j(\rho,z) = \frac{(-1)^{j-1} Qa}{4\pi A_{44}(m_1-m_2)} \left\{\psi(\rho,z_j) - \int_0^{2\pi} \ln\,[\rho^2+a^2-2a\rho\cos(\phi_0)]^{1/2}\,\mathrm{d}\phi_0\right\} \tag{51}$$

Now applying a ρ derivative to the above equation, using the following result from Hanson and Wang (1997)

$$\frac{\partial\psi(\rho,z_j)}{\partial\rho} = 2\pi I_j(0,1;0) + \left\{0, z>0;\ -\frac{4\pi}{\rho}H(a-\rho), z<0\right\} \tag{52}$$

and evaluating the ρ derivative of the integral in eqn (51) as $2\pi H(\rho-a)/\rho$ leads to eqn (45) after some simple manipulations. This verifies the correctness of Hanson and Wang's evaluation for $\partial\psi(\rho,z_j)/\partial\rho$.

The second derivative is given as

$$\frac{\partial^2 F_j(\rho,z)}{\partial\rho^2} = \frac{(-1)^{j+1} Qa}{4\pi A_{44}(m_1-m_2)} \left[\frac{\partial^2\psi(\rho,z_j)}{\partial\rho^2} - \frac{\partial}{\partial\rho}\left\{\frac{2\pi H(\rho-a)}{\rho}\right\}\right] \tag{53}$$

Using $\partial^2\psi(\rho,z_j)/\partial\rho^2$ from Hanson and Wang (1997) will not give the correct answer since they noted that Dirac delta functions were not included in the derivatives of the potential functions since they did not enter into the elastic field. The complete form for $\partial^2\psi(\rho,z_j)/\partial\rho^2$ can be obtained by differentiating eqn (52). The term in brackets will contribute a delta function as

$$\frac{\partial^2 \psi(\rho, z_j)}{\partial \rho^2} = 2\pi I_j(0,0;1) - \frac{2\pi}{\rho} I_j(0,1;0) + \left\{0, z > 0; \frac{4\pi H(a-\rho)}{\rho^2} + \frac{4\pi\delta(\rho-a)}{\rho}, z < 0\right\} \tag{54}$$

Combining the last two equations reveals that eqn (47) is recovered. If the Hanson and Wang (1997) potential in eqn (50) is used along with their result

$$\frac{\partial^2 \psi(\rho, z_j)}{\partial z_j^2} = -2\pi I_j(0,0;1) \tag{55}$$

it is now easy to show that

$$L_j F_j^H(\rho, z) = \frac{(-1)^{j+1} Q a}{A_{44}(m_1 - m_2)} \frac{\delta(\rho - a)}{\rho} [1 - H(z)] \tag{56}$$

Although the potential functions $F_j^H(\rho, z)$ from Hanson and Wang (1997) and $F_j(\rho, z)$ from the transform analysis are different and each satisfies Poisson's equation with a different right hand side [compare eqns (49) and (56)], they will both lead to identical displacement and stress fields. Furthermore although the Dirac delta functions in the derivatives of the potentials were left out in Hanson and Wang (1997), their final results are still correct since these delta functions would have cancelled in the expressions for the elastic field. Of course, they should be added to the derivatives in the Appendices of their paper for complete mathematical correctness of those derivatives.

5. TANGENTIAL LOADING

Now attention is focused on body force loading which is directed parallel to the planes of isotropy, termed tangential loading. Equations (5) and (6) are again used with B_z taken to be zero. A valid solution to eqn (5) is to take the term in brackets equal to zero which, along with the real and imaginary parts of eqn (6), leads to the three equations

$$\frac{m_1}{\gamma_1^2} L_1 F_1 + \frac{m_2}{\gamma_2^2} L_2 F_2 = 0$$

$$\frac{\partial}{\partial x}\left[A_{11} \sum_{j=1}^{2} L_j F_j\right] - \frac{\partial}{\partial y} A_{66} L_3 F_3 = -B_x$$

$$\frac{\partial}{\partial y}\left[A_{11} \sum_{j=1}^{2} L_j F_j\right] + \frac{\partial}{\partial x} A_{66} L_3 F_3 = -B_y \tag{57}$$

For a body force in the x direction only, one can write the potentials as $F_j = \partial M_j/\partial x$, $F_3 = \partial M_3/\partial y$ giving

$$\frac{\partial}{\partial x}\left[\frac{m_1}{\gamma_1^2} L_1 M_1 + \frac{m_2}{\gamma_2^2} L_2 M_2\right] = 0$$

$$\frac{\partial^2}{\partial x^2}\left[A_{11} \sum_{j=1}^{2} L_j M_j\right] - \frac{\partial^2}{\partial y^2} A_{66} L_3 M_3 = -B_x$$

$$\frac{\partial^2}{\partial x \partial y}\left[A_{11} \sum_{j=1}^{2} L_j M_j + A_{66} L_3 M_3\right] = 0 \tag{58}$$

Taking the term in brackets in the last equation equal to zero and then using this result to eliminate $A_{11}\,\Sigma_{j=1}^{2}\,L_jM_j$ from the middle equation, one obtains an equation for M_3 as

$$\Delta L_3 M_3 = L_3(\Delta M_3) = \frac{B_x}{A_{66}} \tag{59}$$

Likewise one can use the last equation to eliminate $A_{66}L_3M_3$ from the middle equation, set the term in brackets in the first equation to zero and then apply the operator Δ to this equation giving

$$\frac{m_1}{\gamma_1^2}L_1(\Delta M_1)+\frac{m_2}{\gamma_2^2}L_2(\Delta M_2) = 0 \quad L_1(\Delta M_1)+L_2(\Delta M_2) = -\frac{B_x}{A_{11}} \tag{60}$$

The above two equations are easily solved as

$$L_j(\Delta M_j) = \frac{(-1)^{j-1}\gamma_j^2 B_x}{A_{44}(m_1-m_2)m_j}, \quad j = 1,2 \tag{61}$$

where the relation $\gamma_1^2\gamma_2^2 = A_{33}/A_{11}$ was used. Equations (59) and (61) are Poisson equations to determine the functions ΔM_j and ΔM_3.

For a concentrated body force at the origin, $B_x = T_x\delta(\rho)\delta(z)/2\pi\rho$ where T_x is the magnitude of the force. Using eqn (19) it is simple to see that

$$\Delta M_3 = -\frac{T_x}{4\pi A_{66}\gamma_3 R_3} = \frac{C_3}{R_3} \tag{62}$$

which is valid since γ_3 is always a positive real number. Even though the elastic field is not axisymmetric, eqn (61) above for ΔM_j is independent of polar angle ϕ for a point force at the origin. Hence the double integral transform in eqn (24) can be used to solve this for any complex γ_j. The result is

$$\Delta M_j = -\frac{(-1)^{j+1}\gamma_j T_x}{4\pi A_{44}(m_1-m_2)m_j}\frac{1}{R_j} = \frac{C_j}{R_j} \tag{63}$$

Obviously $R_j = \sqrt{\rho^2+z_j^2}, j = 1,2,3$ in these formulas.

In the above two equations, the right hand side is independent of polar angle ϕ. Hence it is natural to assume that M_j and M_3 are also independent of ϕ. Thus, the operator Δ becomes an ordinary differential operator in ρ and the solutions are easily written in the form

$$M_j = C_j\left[R_j - z_j\ln\left(\frac{R_j+z_j}{\rho}\right)\right] + U_j(z_j)\ln(\rho) + V_j(z_j) \tag{64}$$

where $j = 1,2,3$ and the last two terms are the homogeneous solutions to the ordinary differential equation with $U_j(z_j)$ and $V_j(z_j)$ being constants with respect to ρ.

Now it is important to make the following observations. From potential theory, $(1/2\pi)\ln(\rho)$ is the two-dimensional full space Green's function and $\Delta(1/2\pi)\ln(\rho)$ corresponds to a point source of unit magnitude at the origin. Also observe that the right hand sides in eqns (62) and (63) do not contain any point sources along the z axis. Furthermore, since V_j is a function of z_j only, it will not contribute to the potentials F_j and F_3 and will be neglected. It would seem reasonable to cancel the $\ln(\rho)$ terms in the above equation by choosing $U_j(z_j) = -C_jz_j$. However this will not eliminate all the point sources since some will also be produced by the term $\ln(R_j+z_j)$.

To establish the behavior of this term, eqns (26) and (30) are used to write

$$\ln(R_j+z_j) = \ln(\rho)+\frac{2}{\pi\gamma_j^2}\int_0^\infty \frac{\sin(\xi z)}{\xi}\int_0^\infty \frac{\eta J_0(\eta\rho)\,\mathrm{d}\eta}{\left(\frac{\eta^2}{\gamma_j^2}+\xi^2\right)}\mathrm{d}\xi \tag{65}$$

If the operator Δ is now applied one obtains

$$\Delta\ln(R_j+z_j) = \Delta\ln(\rho)-\frac{2}{\pi\gamma_j^2}\int_0^\infty \frac{\sin(\xi z)}{\xi}\int_0^\infty \frac{\eta^3 J_0(\eta\rho)\,\mathrm{d}\eta}{\left(\frac{\eta^2}{\gamma_j^2}+\xi^2\right)}\mathrm{d}\xi$$

$$= \frac{\delta(\rho)}{\rho}-\mathrm{sgn}(z)\int_0^\infty \eta\left[1-e^{-\eta\frac{|z|}{\gamma_j}}\right]J_0(\eta\rho)\,\mathrm{d}\eta$$

$$= [1-\mathrm{sgn}(z)]\frac{\delta(\rho)}{\rho}+\frac{z_j}{R_j^3} \tag{66}$$

Now applying the operator Δ to eqn (64) leads to

$$\Delta M_j = \frac{C_j}{R_j}+[U_j(z_j)+C_j z_j\,\mathrm{sgn}(z)]\frac{\delta(\rho)}{\rho} \tag{67}$$

Therefore, the term in brackets above must be taken as zero providing the result

$$M_j = C_j[R_j-z_j\ln(R_j+z_j)+z_j\{1-\mathrm{sgn}(z)\}\ln(\rho)]$$

This can be written using eqn (23) as

$$M_j = C_j[R_j-z_j\ln(R_j+z_j)],\quad z>0 \tag{68}$$

$$= C_j[R_j+z_j\ln(R_j-z_j)],\quad z<0$$

$$= C_j[R_j-|z|_j\ln(R_j+|z|_j)] \tag{69}$$

where $|z|_j = |z|/\gamma_j$. Notice that the above is an even function of z as was the right hand sides of eqns (62) and (63). Substituting for C_j the functions M_j are

$$M_j = \frac{(-1)^{j+1}\gamma_j T_x}{4\pi A_{44}(m_1-m_2)m_j}[|z|_j\ln(R_j+|z|_j)-R_j]$$

$$M_3 = \frac{\gamma_3 T_x}{4\pi A_{44}}[|z|_3\ln(R_3+|z|_3)-R_3] \tag{70}$$

The potentials are thus given as

$$F_j = \frac{(-1)^{j+1}\gamma_j T_x}{4\pi A_{44}(m_1-m_2)m_j}\frac{\partial}{\partial x}[|z|_j\ln(R_j+|z|_j)-R_j]$$

$$F_3 = \frac{\gamma_3 T_x}{4\pi A_{44}}\frac{\partial}{\partial y}[|z|_3\ln(R_3+|z|_3)-R_3] \tag{71}$$

For the body force B_y one can take $F_j = \partial M_j/\partial y$, $F_3 = -\partial M_3/\partial x$ and eqns (57) become

$$\frac{\partial}{\partial y}\left[\frac{m_1}{\gamma_1^2}L_1M_1+\frac{m_2}{\gamma_2^2}L_2M_2\right]=0$$

$$\frac{\partial^2}{\partial x\,\partial y}\left[A_{11}\sum_{j=1}^{2}L_jM_j+A_{66}L_3M_3\right]=0 \tag{72}$$

$$\frac{\partial^2}{\partial y^2}\left[A_{11}\sum_{j=1}^{2}L_jM_j\right]-\frac{\partial^2}{\partial x^2}A_{66}L_3M_3=-B_y \tag{73}$$

Proceeding in the above fashion one can derive an equation for M_3 which is identical to eqn (59) with B_x replaced by B_y. The equations for M_j are identical to eqns (60) with the same replacement. Therefore the potentials are given as

$$F_j=\frac{(-1)^{j+1}\gamma_jT_y}{4\pi A_{44}(m_1-m_2)m_j}\frac{\partial}{\partial y}[|z|_j\ln(R_j+|z|_j)-R_j]$$

$$F_3=-\frac{\gamma_3T_y}{4\pi A_{44}}\frac{\partial}{\partial x}[|z|_3\ln(R_3+|z|_3)-R_3] \tag{74}$$

where T_y is the magnitude of a point body force at the origin in the y direction. Introducing the complex force $T=T_x+iT_y$, these potentials are combined as

$$F_j=\frac{(-1)^{j+1}\gamma_j}{8\pi A_{44}(m_1-m_2)m_j}(T\bar{\Lambda}+\bar{T}\Lambda)[|z|_j\ln(R_j+|z|_j)-R_j]$$

$$F_3=\frac{i\gamma_3}{8\pi A_{44}}(T\bar{\Lambda}-\bar{T}\Lambda)[|z|_3\ln(R_3+|z|_3)-R_3] \tag{75}$$

6. SOLUTION VERIFICATION

The above potentials agree with those given by Fabrikant (1989) for $z>0$. For $z<0$ the potentials differ. Thus it is important to first verify that the potentials above are correct by substituting them directly into the equilibrium eqns (5) and (6). Using eqn (23) it is easy to verify that

$$\mathrm{sgn}(z)\ln(R_j+|z|_j)=\ln(R_j+z_j)+[\mathrm{sgn}(z)-1]\ln(\rho) \tag{76}$$

Now multiplying the above equation by z_j, then subtracting R_j from each side and finally applying the operator Δ gives

$$\Delta[|z|_j\ln(R_j+|z|_j)-R_j]=\Delta[z_j\ln(R_j+z_j)-R_j]+z_j[\mathrm{sgn}(z)-1]\Delta\ln(\rho) \tag{77}$$

where $z_j\,\mathrm{sgn}(z)=|z|_j=|z|/\gamma_j$ has been used. Using eqn (66) allows the right hand side to be evaluated as

$$\Delta[|z|_j\ln(R_j+|z|_j)-R_j]=-\frac{1}{R_j} \tag{78}$$

where the Dirac functions cancel. Furthermore direct differentiation provides

$$\frac{\partial}{\partial z_j}[|z|_j \ln(R_j+|z|_j)-R_j] = \operatorname{sgn}(z)\ln(R_j+|z|_j)$$

$$\begin{aligned}\frac{\partial^2}{\partial z_j^2}[|z|_j \ln(R_j+|z|_j)-R_j] &= \frac{\partial}{\partial z_j}[\operatorname{sgn}(z)\ln(R_j+|z|_j)] \\ &= \ln(R_j+|z|_j)\gamma_j\frac{\partial}{\partial z}\operatorname{sgn}(z)+\operatorname{sgn}(z)\frac{\partial}{\partial z_j}\ln(R_j+|z|_j) \\ &= \frac{1}{R_j}+2\gamma_j\delta(z)\ln(\rho)\end{aligned} \tag{79}$$

Combining the above equations leads to the important result

$$\begin{aligned}L_j[|z|_j \ln(R_j+|z|_j)-R_j] &= \left(\Delta+\frac{\partial^2}{\partial z_j^2}\right)[|z|_j \ln(R_j+|z|_j)-R_j] \\ &= 2\gamma_j\delta(z)\ln(\rho)\end{aligned} \tag{80}$$

Consider now eqn (5). From eqns (75) and (80) it is easy to see the term in brackets sums to zero and hence B_z must be zero. The above results allow eqn (6) to be written as

$$\begin{aligned}B^c &= -\frac{A_{11}}{4\pi A_{44}(m_1-m_2)}\delta(z)(T\Delta+\bar{T}\Lambda^2)\ln(\rho)\sum_1^2\frac{(-1)^{j+1}\gamma_j^2}{m_j}+\frac{1}{4\pi}\delta(z)(T\Delta-\bar{T}\Lambda^2)\ln(\rho) \\ &= \frac{T}{2\pi}\delta(z)\Delta\ln(\rho) = \frac{T}{2\pi}\delta(z)\frac{\delta(\rho)}{\rho} = T\delta(x)\delta(y)\delta(z)\end{aligned} \tag{81}$$

where eqn (17) has been used along with $\gamma_1^2\gamma_2^2 = A_{33}/A_{11}$. Hence the potentials derived in the previous section do correspond to a concentrated tangential body force at the origin.

7. COMPARISON WITH FABRIKANT

The potentials given by Fabrikant are

$$\begin{aligned}F_j^F &= \frac{(-1)^{j+1}\gamma_j}{8\pi A_{44}(m_1-m_2)m_j}(T\bar{\Lambda}+\bar{T}\Lambda)[z_j\ln(R_j+z_j)-R_j] \\ F_3^F &= \frac{i\gamma_3}{8\pi A_{44}}(T\bar{\Lambda}-\bar{T}\Lambda)[z_3\ln(R_3+z_3)-R_3]\end{aligned} \tag{82}$$

where the superscript F denotes his potentials. These agree with the potentials presently derived in eqn (75) for $z > 0$. It is easy to verify that

$$[|z|_j \ln(R_j+|z|_j)-R_j] = [z_j\ln(R_j+z_j)-R_j]-2z_j[1-H(z)]\ln(\rho) \tag{83}$$

Thus the present potentials minus Fabrikant's potentials are termed the "Difference" potentials given by

$$\begin{aligned}F_j^D &= \frac{(-1)^{j+1}\gamma_j}{8\pi A_{44}(m_1-m_2)m_j}(T\bar{\Lambda}+\bar{T}\Lambda)[-2z_j[1-H(z)]\ln(\rho)] \\ F_3^D &= \frac{i\gamma_3}{8\pi A_{44}}(T\bar{\Lambda}-\bar{T}\Lambda)[-2z_3[1-H(z)]\ln(\rho)]\end{aligned} \tag{84}$$

Now for the different potentials to give identical results, these difference potentials should provide a null elastic field as well as equilibrate zero body forces (here we mean zero at every point). If this is true, then there is some degree of arbitrariness in the tangential loading potentials just as exists in the normal loading case.

To determine this, consider first the displacements. From eqn (4) the displacement w is

$$\begin{aligned} w &= \sum_{j=1}^{2} \frac{m_j}{\gamma_j} \frac{\partial}{\partial z_j} F_j^D \\ &= -\frac{1}{4\pi A_{44}(m_1-m_2)}(T\bar{\Lambda}+\bar{T}\Lambda)\ln(\rho)\sum_{j=1}^{2}(-1)^{j+1}\frac{\partial}{\partial z_j}(z_j - z_j H(z)) \\ &= -\frac{1}{4\pi A_{44}(m_1-m_2)}(T\bar{\Lambda}+\bar{T}\Lambda)\ln(\rho)[1-H(z)]\sum_{j=1}^{2}(-1)^{j+1} \\ &= 0 \end{aligned} \tag{85}$$

where the result $z\delta(z) = 0$ was used from generalized function theory. Hence w vanishes at every point. For u^c one can first write

$$\sum_{j=1}^{2} F_j^D + iF_3^D = \frac{T}{2\pi A_{44}}\bar{\Lambda}z\ln(\rho)[1-H(z)] \tag{86}$$

thus, giving

$$\begin{aligned} u^c &= \Lambda\left(\sum_{j=1}^{2} F_j^D + iF_3^D\right) \\ &= \frac{T}{2\pi A_{44}}\Delta z\ln(\rho)[1-H(z)] \\ &= \frac{T}{A_{44}} z[1-H(z)]\frac{\delta(\rho)}{2\pi\rho} \\ &= \frac{T}{A_{44}} z[1-H(z)]\delta(x)\delta(y) \end{aligned} \tag{87}$$

This illustrates that the tangential displacements are zero everywhere except along the negative z axis where they are Dirac singular. This clearly illustrates that the different form to the potentials leads to different tangential displacements and they are not equivalent.

On the surface this last result may seem like a trivial detail since, for all intents and purposes, it is essentially zero. Indeed if all one was interested in was the elastic field for a point tangential force, the different potentials would produce the same displacements since the Dirac term above would not come into play. However, the value of the point force solution rests in its use as a Green's function to generate solutions to distributed loading. Hence if one integrates the point force solution over a line (straight or curved), an area or a volume, the additional Dirac term above (which should not be there) may then come into play.

It is also interesting to see the body force field created by the difference potentials. It is easy to obtain by direct differentiation that

$$L_j[-2z_j[1-H(z)]\ln(\rho)] = \left\{2\gamma_j\delta(z)\ln(\rho) - 2z_j[1-H(z)]\frac{\delta(\rho)}{\rho}\right\} \tag{88}$$

Hence the body forces from the difference potentials can be obtained from eqns (5) and (6) as

$$B_z^D = \frac{A_{11}[1-H(z)]}{A_{44}(m_1-m_2)}[T_x\delta'(x)\delta(y)+T_y\delta(x)\delta'(y)](\gamma_2^2-\gamma_1^2) \tag{89}$$

$$B^{cD} = T\delta(x)\delta(y)\delta(z)-\frac{z}{2}[1-H(z)]\left\{\frac{(A_{11}+A_{66})}{A_{44}}T\Delta+\frac{(A_{11}-A_{66})}{A_{44}}\bar{T}\Lambda^2\right\}\delta(x)\delta(y) \tag{90}$$

Considering Fabrikant's potentials and using the result

$$L_j[z_j\ln(R_j+z_j)-R_j] = z_j[1-\mathrm{sgn}(z)]\frac{\delta(\rho)}{\rho}$$

one can derive the following expressions from eqns (5) and (6)

$$B_z^F = -\frac{A_{11}[1-\mathrm{sgn}(z)]}{2A_{44}(m_1-m_2)}[T_x\delta'(x)\delta(y)+T_y\delta(x)\delta'(y)](\gamma_2^2-\gamma_1^2) \tag{91}$$

$$B^{cF} = \frac{z}{4}[1-\mathrm{sgn}(z)]\left\{\frac{(A_{11}+A_{66})}{A_{44}}T\Delta+\frac{(A_{11}-A_{66})}{A_{44}}\bar{T}\Lambda^2\right\}\delta(x)\delta(y) \tag{92}$$

It is now easy to see that $B_z^D+B_z^F = 0$, $B^{cD}+B^{cF} = T\delta(x)\delta(y)\delta(z)$. Hence the difference potentials plus Fabrikant's potentials together add up to the correct body force as we know they should. This sum is exactly the presently derived potentials. It is interesting to observe that the Fabrikant potentials do not include a point tangential body force term, only derivatives of a point body force, even though they give the correct elastic field for $z > 0$.

8. RING TANGENTIAL LOADING

Now consider a uni-directional ring tangential load applied in the $z = 0$ plane with radius a. The forces per unit circumferential length in the x and y directions are denoted as S_x and S_y. The complex tangential body forces are thus, given as $B^c = Sa\delta(\rho-a)\delta(z)/\rho$ where $S = S_x+iS_y$. The solutions for B_x and B_y were found separately. For B_x eqns (59) and (61) allowed ΔM_j to be evaluated (using the double integral transform) as

$$\Delta M_j = \frac{(-1)^j\gamma_j S_x a}{2A_{44}(m_1-m_2)m_j}\int_0^\infty J_0(\eta a)J_0(\eta\rho)\,e^{-\eta\frac{|z|}{\gamma_j}}\,\mathrm{d}\eta$$

$$\Delta M_3 = -\frac{S_x a}{2A_{66}\gamma_3}\int_0^\infty J_0(\eta a)J_0(\eta\rho)\,e^{-\eta\frac{|z|}{\gamma_3}}\,\mathrm{d}\eta \tag{93}$$

The solution for M_j can then be found as

$$M_j = \frac{(-1)^{j+1}\gamma_j S_x a}{2A_{44}(m_1-m_2)m_j}\int_0^\infty \frac{1}{\eta^2}J_0(\eta a)J_0(\eta\rho)\,e^{-\eta\frac{|z|}{\gamma_j}}\,\mathrm{d}\eta$$

$$M_3 = \frac{S_x a}{2A_{66}\gamma_3}\int_0^\infty \frac{1}{\eta^2}J_0(\eta a)J_0(\eta\rho)\,e^{-\eta\frac{|z|}{\gamma_3}}\,\mathrm{d}\eta \tag{94}$$

Although these integrals satisfy the differential equations, they are not formally convergent. The results for S_y are as given above with S_x replaced by S_y. For a body force in the x direction the potentials are given as $F_j = \partial M_j/\partial x$, $F_3 = \partial M_3/\partial y$ while for a body force in the

y direction $F_j = \partial M_j/\partial y$, $F_3 = -\partial M_3/\partial x$. Evaluating the potentials and then combining the results allows the final form to be given as

$$F_j = \frac{(-1)^{j+1} a\gamma_j [S e^{-i\phi} + \bar{S} e^{i\phi}]}{8\pi A_{44}(m_1 - m_2) m_j} \{-2\pi I_{|j|}(0,1;-1)\}$$

$$F_3 = \frac{i a\gamma_3 [S e^{-i\phi} - \bar{S} e^{i\phi}]}{8\pi A_{44}} \{-2\pi I_{|3|}(0,1;-1)\} \tag{95}$$

These agree with the potentials derived by Hanson and Wang (1997) [eqn (70) of their paper] for $z > 0$. For $z < 0$, they have an extra discontinuous term since they used the point force potentials from Fabrikant (1989).

Now compare the displacement field. From the present results the displacement w becomes

$$w = \frac{a[S_x \cos\phi + S_y \sin\phi]}{2A_{44}(m_1 - m_2)} \sum_{j=1}^{2} (-1)^{j+1} \operatorname{sgn}(z) I_{|j|}(0,1;0) \tag{96}$$

which should be an odd function of z as it correctly is. Now using eqn (43) one has

$$\sum_{j=1}^{2} (-1)^{j+1} \operatorname{sgn}(z) I_{|j|}(0,1;0) = \sum_{j=1}^{2} (-1)^{j+1} \left[I_j(0,1;0) + \left\{ 0, z > 0; -\frac{2}{\rho}, z < 0 \right\} \right]$$

$$= \sum_{j=1}^{2} (-1)^{j+1} I_j(0,1;0) \tag{97}$$

since the extra term cancels in the summation. Hence the expression for w derived by Hanson and Wang (1997) will give the correct result although eqn (96) is preferred as being correct. This result was expected since the difference potentials gave a zero w.

Now consider the tangential displacement u^c. It can be shown that the presently derived potentials lead to

$$u^c = -\frac{a}{4A_{44}(m_1 - m_2)} \sum_{j=1}^{2} \frac{(-1)^{j+1}\gamma_j}{m_j} \left\{ S I_{|j|}(0,0;0) + \bar{S} e^{2i\phi} \left[I_{|j|}(0,0;0) - \frac{2}{\rho} I_{|j|}(0,1;-1) \right] \right\}$$

$$+ \frac{a\gamma_3}{4A_{44}} \left\{ S I_{|3|}(0,0;0) - \bar{S} e^{2i\phi} \left[I_{|3|}(0,0;0) - \frac{2}{\rho} I_{|3|}(0,1;-1) \right] \right\} \tag{98}$$

This expression is an even function of z which is correct on physical considerations. The results in eqn (74) of Hanson and Wang (1997) are

$$u^c = -\frac{a}{4A_{44}(m_1 - m_2)} \sum_{j=1}^{2} \frac{(-1)^{j+1}\gamma_j}{m_j} \left\{ S I_j(0,0;0) + \bar{S} e^{2i\phi} \left[I_j(0,0;0) - \frac{2}{\rho} I_j(0,1;-1) \right] \right\}$$

$$+ \frac{a\gamma_3}{4A_{44}} \left\{ S I_3(0,0;0) - \bar{S} e^{2i\phi} \left[I_3(0,0;0) - \frac{2}{\rho} I_3(0,1;-1) \right] \right\} \tag{99}$$

The results obviously agree for $z > 0$. Since $I_j(0,0;0)$ is a function of z^2, $I_j(0,0;0) = I_{|j|}(0,0;0)$ also for $z < 0$. The term $I(0,1;-1)$ is given as

$$I(0,1;-1) = -\frac{z}{\rho} + \frac{2}{\pi\rho l_2}\{l_2^2\mathbf{E}(k) - (l_2^2-\rho^2)\mathbf{F}(k) + z^2\mathbf{\Pi}(n,k)\} \tag{100}$$

where $\mathbf{F}(k)$ and $\mathbf{E}(k)$ are complete elliptic integrals of the first and second kinds. The first term is an odd function of z while the other terms are even functions of z. Substituting z_j for z, it is easy to show that this odd function cancels out of the u^c expression in eqn (99) above and thus the two different forms again give identical results.

A final point is now considered. The results reported by Hanson and Wang (1997) for tangential loading in Appendix B and C of their paper evaluated the derivatives of the potential functions based on Fabrikant's point force solutions. For the region $z < 0$ some of the evaluations resulted in a discontinuous function. A differentiation of these in the ρ direction should have also included a Dirac function in some of the terms. They noted that these delta function terms were neglected since they would not directly contribute to the elastic field. This is fortuitous since, based on the correct potentials derived in the present paper given in eqn (75), the discontinuous terms should not have been there to start with and thus the delta functions would also not occur.

The consistency of all this analysis will be illustrated by considering the expression for u^c. From eqn (71) of Hanson and Wang (1997) u^c is given as

$$u^c = \frac{a}{8\pi A_{44}(m_1-m_2)}\sum_{j=1}^{2}\frac{(-1)^{j+1}\gamma_j}{m_j}[S\Delta+\bar{S}\Lambda^2]\Gamma(\rho,z_j) - \frac{a\gamma_3}{8\pi A_{44}}[S\Delta-\bar{S}\Lambda^2]\Gamma(\rho,z_3) \tag{101}$$

From Appendix B of their paper one has

$$\Delta\Gamma(\rho,z_j) = -2\pi I_j(0,0;0) + \frac{4\pi z_j}{\rho}\delta(\rho-a)[1-H(z)] \tag{102}$$

$$\Lambda^2\Gamma(\rho,z_j) = e^{2i\phi}\left\{-2\pi\left(I_j(0,0;0) - \frac{2}{\rho}I_j(0,1;-1)\right) + \frac{4\pi z_j}{\rho}\delta(\rho-a)[1-H(z)] + \frac{8\pi z_j}{\rho^2}H(a-\rho)[1-H(z)]\right\} \tag{103}$$

where the Dirac terms they neglected have now been included. We have already shown that the terms containing $I_j(\mu,\nu;\lambda)$ give the correct elastic field and they showed that the discontinuous term (the last term in the above equation) will cancel out the summation and not contribute. If we consider only the Dirac terms the displacement u^c from eqn (101) becomes

$$u^c = -\frac{Saz}{A_{44}}\frac{\delta(\rho-a)}{\rho}[1-H(z)] \tag{104}$$

and thus, the Dirac terms do not cancel out of the elastic field. The present correct potentials do not give any delta functions. Hence if we add the effect of the difference potentials to the above result, it should sum to zero. The displacement resulting from the difference potentials is given in eqn (87) for a point force at the origin. For a ring load let $T \to Sad\phi_0$, $\delta(x)\delta(y) \to (1/\rho)\delta(\rho-a)\delta(\phi-\phi_0)$ and integrate over $0 < \phi_0 < 2\pi$ giving

$$u^c = \frac{Saz}{A_{44}}\frac{\delta(\rho-a)}{\rho}[1-H(z)] \tag{105}$$

which is just a negative sign difference and the two results will sum to zero.

9. CONCLUSION

The present analysis has derived a potential function solution to the equilibrium equations of linear elasticity for a transversely isotropic body. Here the significant aspect was the inclusion of body force terms. It is obvious from eqns (5) and (6) that the potentials can be defined to within an arbitrary harmonic function H_j which satisfies $L_jH_j = 0$. For an infinite geometry, a function which is harmonic, gives rise to no source terms and whose derivatives vanish at infinity must be zero. However it was shown that for a body force perpendicular to the isotropic planes, some freedom still exists in the choice of the potential that will lead to the correct elastic field. In fact, the potential can differ by an arbitrary function of the polar radius without affecting the elastic field or the body force distribution. For tangential loading such freedom may not exist, although the present analysis has not tried to prove this conclusion. Present analysis has derived the correct potentials for point loading parallel to the isotropic planes. For $z > 0$ the potentials given by Fabrikant (1989) lead to the correct elastic field. For $z < 0$ those potentials will produce extraneous mathematical terms when used as a Green's function for distributed loading cases as found by Hanson and Wang (1997). In regards to this recent ring loading analysis, it can now be stated that the final results giving the elastic fields in their paper for the various cases are correct. This conclusion could also have been conjectured without proof since their expressions satisfied the correct symmetry or anti-symmetry conditions for each problem. The discontinuous terms should not have been there to begin with however they cancelled out the elastic fields. The Dirac terms that were left out of their derivatives should likewise not have been there anyway.

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